IT 971: Probability Theory

Problem Set 2

1. If $E[X_n] = 1$ and $E[X_n^2]$ is bounded in n, then

$$P\left\{\overline{\lim_{n\to\infty}} X_n \geq 1\right\} > 0$$

   This is due to Kochen and Stone. Truncate X_n at A to Y_n with A so large that $E[Y_n] > 1 - \epsilon$ for all n.

2. For any sequence of random variables $\{X_n\}$, and any $p \geq 1$;

$$X_n \to 0 \text{ almost surely } \Rightarrow \frac{S_n}{n} \to 0 \text{ almost surely},$$

$$X_n \to 0 \text{ in } L^p \Rightarrow \frac{S_n}{n} \to 0 \text{ in } L^p.$$

   Show the second result is false for $p < 1$.

3. For arbitrary $\{X_n\}$, if

$$\sum_n E[|X_n|] < \infty$$

   then $\sum_n X_n$ converges absolutely almost surely

4. Let $E[X_1] = 0$ and $\{c_n\}$ be a bounded sequence of real numbers. Then

$$\frac{1}{n}\sum_{j=1}^{n} c_j X_j \to 0 \text{ almost surely}$$

   [Hint: Truncate X_n at n.]

5. We have $\frac{S_n}{n} \to 0$ almost surely if and only if the following two conditions are satisfied:

$$(i)\frac{S_n}{n} \to 0 \text{ in pr.},$$

$$(ii)\frac{S_{2^n}}{2^n} \to 0 \text{ almost surely};$$

   an alternative set of conditions is (i) and (ii') $\forall \epsilon > 0 : \sum_n P(|S_{2^{n+1}} - S_{2^n}| > 2^n\epsilon) < \infty$.

6. If f is a ch.f., and G a d.f. with $G(0-) = 0$, then the following functions are all ch.f.'s:

$$\int_0^1 f(ut)du \ , \ \int_0^\infty f(ut)e^{-u}du \ , \ \int_0^\infty e^{-|t|u}dG(u) \ ,$$

$$\int_0^\infty e^{-t^2u}dG(u) \ , \ \int_0^\infty f(ut)dG(u).$$

7. Show that except for a Borel set of measure 0, every number in [0,1] is simply normal. (See Lecture 8, page 5)

8. Let $\{X_n\}$ be a sequence of independent random variables with distribution functions $\{F_n\}$. Let

$$S_n = \sum_{j=1}^{n} X_j.$$

Let $\{b_n\}$ be a sequence of real numbers $\ni b_n \nearrow \infty$ and

$$i) \sum_{j=1}^{n} \int_{|X|>b_n} dF_j(x) = o(1)$$

$$ii) \frac{1}{b_n^2} \sum_{j=1}^{n} \int_{|X|>b} X^2 dF_j(x) = o(1)$$

Then, if

$$a_n = \sum_{j=1}^{n} \int_{|x|\le b_n} X dF_j(x)$$

we have $\frac{S_n - a_n}{b_n} \to 0$ in probability.

Moreover suppose $\exists \lambda > 0 \ni F_n(0) \ge \lambda \; \forall n$ and

$$1 - F_n(0-) \ge \lambda \; \forall n \text{ (i.e. } F_n(0-) \le 1 - \lambda \; \forall n)$$

then if $\frac{S_n - a_n}{b_n} \to 0$ in probability

$i)$ and $ii)$ must hold. Prove the converse. (See Lecture 8, page 9)