Lecture 12

Central Limit Theorem(s) and Related Topics

Earlier we had

CL Theorem: If F has mean μ and finite variance $\sigma^2 > 0$, then

$$\frac{S_n - \mu n}{\sigma\sqrt{n}} \to \Phi \text{ in distribution}$$

where Φ is the normal distribution with mean 0 and variance 1.

We now want to consider more general formulations

$$\frac{S_n - a_n}{b_n} = \left(\sum_{j=1}^{n} \frac{X_j}{b_n}\right) - \frac{a_n}{b_n}$$

Consider the formulation: For each $n \geq 1$, let there be k_n random variables $\{X_{nj} : 1 \leq j \leq k_n\}$ with $k_n \to \infty$ as $n \to \infty$.

$$\circledast \quad \begin{array}{l} X_{11}, X_{12}, \cdots, X_{1k_1} \\ X_{21}, X_{22}, \cdots, X_{2k_2} \\ \vdots \\ X_{n1}, X_{n2}, \cdots, X_{nk_n} \end{array}$$

Let $F_{n,j}$ be the distribution function of X_{nj} and Ψ_{nj} be the characteristic function of X_{nj}.

In the particular case, $k_n = n$ for each n yields a triangular array and $X_{nj} = X_j$, then the situation reduces to the initial segments of the $\{X_j, j \geq 1\}$.

We assume the random variables in each row are independent, but those in the different rows may be dependent

Notation

$$E[X_{nj}] = \alpha_{nj},\ \sigma^2(X_{nj}) = \sigma_{nj}^2$$

$$E[S_n] = \sum_{j=1}^{k_n} \alpha_{nj} = \alpha_n,\ \sigma^2(S_n) = \sum_{j=1}^{k_n} \sigma_{nj}^2 = s_n^2$$

$$E[|X_{nj}|^3] = \gamma_{nj},\ \Gamma_n = \sum_{j=1}^{k_n} \gamma_{nj}.$$

In our previous special case $X_{nj} = \frac{X_j}{b_n}$

$$\Rightarrow \sigma^2(X_{nj}) = \frac{\sigma^2(X_j)}{b_n^2}$$

If we take $b_n = s_n$,

$$\sum_{j=1}^{n} \sigma^2(X_{nj}) = 1.$$

As before we may consider $X_{nj} - \alpha_{nj}$ instead of X_{nj}.

Then this is equivalent to supposing the means are all 0, i.e. $\alpha_{nj} = 0 \;\forall\; j, \forall\; n$.

In dealing with CLT (classical case) we are concerned with

$$\frac{S_n}{\sigma\sqrt{n}} = \sum_{j=1}^{n} \frac{X_j}{\sigma\sqrt{n}} \text{ as } n \to \infty.$$

notice that the individual components of the sum become increasingly negligible as $n \to \infty$. In the general case, we also want the same:

$$S_n = \sum_{j=1}^{k_n} X_{nj},\; X_{nj}$$

are negligible compared to S_n.

Meaning of negligible - $\forall \epsilon > 0$

$$a) \lim_{n\to\infty} P[|X_{nj}| > \epsilon] = 0$$

$$b) \lim_{n\to\infty} \max_{1\le j\le k_n} P[|X_{nj}| > \epsilon] = 0$$

$$c) \lim P\left[\max_{1\le j\le k_n} |X_{nj}| > \epsilon\right] = 0.$$

$$d) \lim_{n\to\infty} \sum_{j=1}^{k_n} P[|X_{nj}| > \epsilon] = 0.$$

In general, $d) \Rightarrow c) \Rightarrow b) \Rightarrow a)$.

Theorem: $*$ has condition $b)$ iff

$$\lim_{n\to\infty} \max_{1\le j\le k_n} |\Psi_{nj}(t) - 1| = 0.$$

Lemma: Let $\{\theta_{nj} : 1 \leq j \leq k_n, 1 \leq n\}$ be a doubly indexed array of complex numbers satisfying as $n \to \infty$.

$$i) \max_{1 \leq j \leq k_n} |\theta_{nj}| \to 0$$

$$ii) \sum_{j=1}^{k_n} |\theta_{nj}| \leq M < \infty \text{ where } M \text{ does not depend on } n.$$

$$iii) \sum_{j=1}^{k_n} \theta_{nj} \to \theta \text{ where } \theta \text{ is a finite complex number.}$$

$$\Rightarrow \prod_{j=1}^{n} (1 + \theta_{nj}) \to e^{\theta}$$

Theorem: Assume

$$\circledast\circledast \quad \begin{array}{l} E[X_{nj}] = \alpha_{nj},\ \sigma^2(X_{nj}) = \sigma_{nj}^2 \\ E[S_n] = \alpha_n,\ \sigma^2(S_n) = s_n^2 \\ E[|X_{nj}|^3] = \gamma_{nj},\ \Gamma_n = \sum_{j=1}^{k_n} \gamma_{nj} \\ \sum_{j=1}^{k_n} \sigma^2(X_{nj}) \end{array}$$

If $\Gamma_n \to 0$ as $n \to \infty$, then S_n converges in distribution to Φ.

Corollary: Without supposing that $E[X_{nj}] = 0$, suppose for each n and j there is a finite constant $M_{nj} \ni |X_{nj}| \leq M_{nj}$ a.s.

and

$$\max_{1 \leq j \leq k_n} M_{nj} \to 0 \text{ as } n \to \infty$$

then

$$S_n - ES_n \text{ converges in distribution to } \Phi.$$

Lindeberg-Feller Central Limit Theorem:
Assume $\sigma_{nj}^2 < \infty$ for each n and j and assume $\circledast\circledast$ holds.

As $n \to \infty$

$$i) S_n \text{ converges in distribution to } \Phi$$

$$ii)\ b) \text{ holds}$$

iff for each $\eta > 0$

$$iii) \sum_{j=1}^{k_n} \int_{|X| > \eta} X^2 dF_{nj}(x) \to 0$$

$iii)$ is called the Lindeberg's condition.

Equivalently

$$\sum_{j=1}^{k_n} \int_{|X|\leq \eta} X^2 dF_{n,j}(x) \to 1.$$

Lévy's Central Limit Theorem:
Let $\{X_j : j \geq 1\}$ be independent random variables having common distribution function F. Let

$$S_n = \sum_{j=1}^{n} X_j. \ \exists \text{ constants } a_n \text{ and } b_n > 0, b_n \to \infty$$

$\ni \frac{S_n - a_n}{b_n}$ converges in distribution to Φ

iff as $y \to \infty$

$$y^2 \int_{|x|>y} dF(x) = o\left(\int_{|x|\leq y} x^2 dF(x)\right)$$

m**-dependence**.
Let $\{X_n, n \geq 1\}$ be a sequence of random variables.

Let $\mathcal{A}_n$ be the Borel field generated by $\{X_k : 1 \leq k \leq n\}$ and $\mathcal{A}_n^*$ be the Borel field generated by $\{X_k : n < k < \infty\}$.

The sequence is m-dependent if there exist an integer $m \geq 0 \ni \forall\, n$, $\mathcal{A}_n$ and $\mathcal{A}_{n+m}^*$ are independent.

Theorem: CLT for m-dependent sequences.

Suppose $\{X_n\}$ is a sequence of m-dependent, uniformly bounded random variables $\ni$

$$\frac{\sigma(S_n)}{n^{1/3}} \to \infty \text{ as } n \to \infty$$

Then $\frac{S_n - E(S_n)}{\sigma(S_n)}$ converges in distribution to Φ.

Remainder terms –

Suppose Y_n is a sequence of random variables with distribution function F_n, $\ni$

$$F_n(x) \to \Phi(x).$$

What can one say about $|F_n(x) - \Phi(x)|$?

Theorem: Assuming $**$ holds for $\circledast$ and that γ_{nj} is finite $\forall\ n \geq j$. If $\Gamma_n \to 0$, then $\exists$ a constant $A_0 \ni$

$$\sup_x |F_n(x) - \Phi(x)| \leq A_0 \Gamma_n,$$

where F_n is the distribution function of S_n.

Law of the Iterated Logarithm

Theorem: LIL (Kolmogorov)

Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with

$$S_n = \sum_{j=1}^{n} X_j.$$

Suppose $E[X_n] = 0\ \forall n$ and

$$\sup_\omega |X_n(\omega)| = o\left(\frac{s_n}{\sqrt{\log\log s_n}}\right)$$

where $s_n^2 = \sigma^2(S_n)$,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log\log s_n}} = 1.$$

Wintner also shows conclusion holds if $\{X_n\}$ are iid. with finite 2nd moment.

Theorem: (Feller): For any increasing sequence c_n.

$$P[S_n(\omega) > s_n c_n \text{ i.o.}] = \begin{cases} 0 \\ 1 \end{cases}$$

according as the series

$$\sum_{n=1}^{\infty} \frac{c_n}{n} e^{-\frac{c_n^2}{2}} \begin{cases} < \infty \\ = \infty \end{cases}$$

Example: Consider

$$I_j(x) = \begin{cases} 1 & X_j \leq x \\ 0 & X_j > x. \end{cases}$$

Then $F_n(x) = \frac{1}{n} \sum I_j(x)$ is the empirical distribution function

$$S_n = \sum I_j(x).\ E[S_n] = nP[X \leq x] = nF(x).$$
$$EI_j(x) = P[X_j \leq x] = p.$$

Then $ES_n = np$
$E(S_n - np) = 0.$

$$var(S_n - np) = var\left(\sum (I_j - p)\right)^2$$
$$= \sum_1^n E[I_j - p]^2$$
$$= \sum_1^n [(-p)^2 \cdot (1-p) + (1-p)^2(p)]$$
$$= \sum_1^n p^2(1-p) + (1-p)^2 p$$
$$= \sum_1^n p(1-p)[p + 1 - p] = \sum_1^n p(1-p) = np(1-p)$$

Thus $s_n^2 = np(1-p)$. $\{X_j\}$ are iid.

$EX_j^2 = p(1-p) < \infty$.

By Wintner version
$$\limsup_{n\to\infty} \frac{n[F_n(x) - F(x)]}{\sqrt{2np(1-p)\log\log np(1-p)}} = 1.$$
$$\limsup_{n\to\infty} \frac{\sqrt{n}[F_n(x) - F(x)]}{\sqrt{2F(x)(1-F(x))\log\log nF(x)(1-F(x))}} = 1.$$
exact rate of convergence for empirical dist.

Infinite Divisibility

Weak Law of Large Numbers. concerned with convergence in distribution of sums of independent random variables to a degenerate dist (all mass at μ).

Central Limit Theorem. concerned with convergence in distribution of sums of independent random variables to a standard normal.

$\psi(t) = e^{\mu it}$ for degenerate case

$\psi(t) = e^{\mu it - \frac{t^2}{2\sigma^2}}$ for normal case

Question: Are there any other limiting distribution functions?

Consider Poisson case $\psi(t) = e^{\lambda\left(e^{it} - 1\right)}$, $\lambda > 0$

Symmetric stable dist of exponent α,
$\psi(t) = e^{-c|t|^\alpha}$, $0 < \alpha < 2$, $c > 0$.

$\alpha = 1$ is the Cauchy distribution.
$\alpha = 2$ is the Normal distribution.

Notice these are all exponentials. Moreover

In each case the nth root is also a characteristic function

$$e^{\frac{\mu}{n}it},\ e^{\frac{\mu}{n}it - \frac{t^2}{2n\sigma^2}},\ e^{\frac{\lambda}{n}\left(e^{it}-1\right)},\ e^{-\frac{c}{n}|t|^\alpha}$$

which is of the same type as the original characteristic function.

Definition: A characteristic function ψ is infinitely divisible iff for each $n \geq 1$ $\exists$ ψ_n ∍

$$\psi = (\psi_n)^n$$

In terms of distribution functions

$$F = \underbrace{F_n * F_n * \cdots * F_n}_{n \text{ factors}} \stackrel{d}{=} F_n^{n*}$$

Theorem: An infinitely divisible characteristic function never vanishes.