Coordinate Transformations

A coordinate transformation is a useful device in ordinary geometry. The same idea can be established in d-dimensional space. Consider a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ and a point $\boldsymbol{y} = (y_1, y_2, \ldots, y_d)$. Then a **general coordinate transformation** is given by

$$y_i = \sum_{j=1}^{d} l_{ij}\, x_j \, + a_i,\ \ i = 1, \ldots, d.$$

This transformation **translates** the x coordinate system with origin $\mathbf{0}$ to the y coordinate system with origin at $\boldsymbol{a} = (a_1, \ldots, a_d)$. The transformation also **rotates** the axes by the rotation matrix $\boldsymbol{L} = [\, l_{ij} \,]$.

Consider only the rotation part of this transformation

$$y_i = \sum_{j=1}^{d} l_{ij}\, x_j.$$

If

$$\sum_{j=1}^{d} l_{ij}\, l_{jk} = \begin{cases} 1,\ i = k \\ 0,\ i \neq k \end{cases}$$

then the rotation is said to be **orthogonal.**

Actually it could also be said to be **orthonormal.**

Since $\boldsymbol{L} = [\, l_{ij} \,]$, $\boldsymbol{L}^{\mathsf{T}} = [\, l_{ji} \,]$ where the **T** indicates transpose. Then the orthogonality condition becomes $\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} = \boldsymbol{I}$, where the $\boldsymbol{I}$ is the identity matrix. Thus $\boldsymbol{L}^{\mathsf{T}} = \boldsymbol{L}^{-1}$. This also implies $1 = |\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}| = |\boldsymbol{L}||\boldsymbol{L}^{\mathsf{T}}|$.

Since $|\boldsymbol{L}| = |\boldsymbol{L}^{\mathsf{T}}|$, we have that $|\boldsymbol{L}|^{\mathbf{2}} = 1$. $\boldsymbol{L}^{-\mathbf{1}}$ is also an orthogonal transform, so that $\boldsymbol{L}$ is said to be **biorthogonal**. Notice if $\boldsymbol{y} = \boldsymbol{L}\boldsymbol{x}$, then $\boldsymbol{x} = \boldsymbol{L}^{-\mathbf{1}}\boldsymbol{y} = \boldsymbol{L}^{\mathsf{T}}\boldsymbol{y}$.

Polar Coordinates in d-Space

Polar coordinates can be generalized to d-dimensions.

Recall for $d = 2$,

$$x_1 = r\cos(\theta)$$
$$x_2 = r\sin(\theta)$$

with $r \geq 0$, $0 \leq \theta \leq 2\pi$ and $x_1^2 + x_2^2 = r^2$.

The Jacobian of this transformation is given by

$$J = \left|\frac{\partial(x_1,x_2)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{vmatrix} = r.$$

For $d = 3$,

$$x_1 = r\cos(\theta_1)\cos(\theta_2) = r\, c_1 c_2$$
$$x_2 = r\cos(\theta_1)\sin(\theta_2) = r\, c_1 s_2$$
$$x_3 = r\sin(\theta_1) \ = r\, s_1$$

where $r \geq 0,\ -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2},\quad 0 \leq \theta_2 \leq 2\pi$ and $x_1^2 + x_2^2 + x_3^2 = r^2$.

Here the Jacobian of the transformation is

$$J = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(r,\theta_1,\theta_2)}\right| = \begin{vmatrix} c_1c_2 & c_1s_2 & s_1 \\ -r\,s_1c_2 & -r\,s_1s_2 & r\,c_1 \\ -r\,c_1s_2 & r\,c_1c_2 & 0 \end{vmatrix} = r^2\cos(\theta_1).$$

For the general d-dimensional case,

$$\begin{aligned} x_1 &= r\ c_1c_2\cdots c_{d-2}c_{d-1} \\ x_2 &= r\ c_1c_2\cdots c_{d-1}s_{d-1} \\ x_3 &= r\ c_1c_2\cdots c_{d-3}s_{d-2} \\ &\vdots \\ x_j &= r\ c_1c_2\cdots c_{d-j}s_{d-j+1} \\ &\vdots \\ x_d &= r\ s_1 \end{aligned}$$

where $r \geq 0,\ -\frac{\pi}{2} \leq \theta_{d-j} \leq \frac{\pi}{2},\ j = 2, \ldots, d-1,$
$0 \leq \theta_{d-1} \leq 2\pi$ and $\sum_{j=1}^{d} x_j^2 = r^2$.

Here the Jacobian of the transformation is

$$J = \left|\frac{\partial(x_1, x_2, \ldots, x_d)}{\partial(r, \theta_1, \theta_2, \ldots, \theta_{d-1})}\right| = r^{d-1} c_1^{d-2} c_2^{d-3} \cdots c_{d-2}.$$

Equation of a Flat

In d dimensions, the general equation of a $(d-1)$-flat is

$$a_1 y_1 + a_2 y_2 + \cdots + a_d y_d = k$$

where $a_1, a_2, \ldots, a_d$ and k are $d+1$ constants.

There are $(d-1)$ degrees of freedom since if $y_1, \ldots, y_{d-1},$ then y_d is determined. If this $(d-1)$-plane passes through d points with coordinates $\boldsymbol{x_i} = (x_{1i},\ x_{2i}, \ldots, x_{di}), i = 1, \ldots, d$, then these points satisfy the d equations

$$a_1 x_{1i} + a_2 x_{2i} + \cdots + a_d x_{di} = k,\ i = 1, \ldots, d.$$

That is

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_d & 1 \\ x_{11} & x_{21} & \cdots & x_{d1} & 1 \\ x_{12} & x_{22} & \cdots & x_{d2} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1d} & x_{2d} & \cdots & x_{dd} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \\ -k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Since there are $d+1$ non-zero constants, the equation must satisfy

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_d & 1 \\ x_{11} & x_{21} & \cdots & x_{d1} & 1 \\ x_{12} & x_{22} & \cdots & x_{d2} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1d} & x_{2d} & \cdots & x_{dd} & 1 \end{vmatrix} = 0. \qquad (2.8)$$

If $[\, x_{ij} \,]$ is not of rank d, the points are not linearly independent. This implies the $(d-1)$-plane is not uniquely determined.

Angles

Angle Between Any Two Lines

Any two intersecting straight lines determine a 2-plane in which both lines lie. In that 2-plane, there are two angles between the lines, say θ and $180° - \theta$. There is one unique angle, say θ, between 0° and 90° inclusive.

If two lines, say $\mathcal{L}_1$ and $\mathcal{L}_2$, do not intersect, we can choose a point on one, say $\mathcal{L}_1$, and draw a line through that point which is parallel to the second line $\mathcal{L}_2$. This defines an angle, θ, as above which is independent of the point we choose on $\mathcal{L}_1$. Thus we can define an angle between any two lines in d-dimensional space.

Angle Between a Line and a $(d-1)$-Flat

We can also determine an angle between a line and a $(d-1)$-flat. A general line in d-space is given according to (2.2) by

$$\frac{y_1 - x_1}{l_1} = \frac{y_2 - x_2}{l_2} = \cdots = \frac{y_d - x_d}{l_d}$$

where $(x_1, x_2, \ldots, x_d)$ is a fixed point on the line. Hence a line parallel to it through the origin would be

$$\frac{y_1}{l_1} = \frac{y_2}{l_2} = \cdots = \frac{y_d}{l_d}.$$

The l_i are indeterminate in the sense that we could rescale all by the same amount and not change the line. Hence we may require

$$\sum_{i=1}^{d} l_i^2 = 1.$$

Now consider a $(d-1)$-plane given by $y_1 = a_1$. This plane will meet the line in a point since $S_1 \cap S_{d-1} = S_{d-(d-1+1)} = S_0$, a point.

Now since $\frac{a_1}{l_1} = \frac{y_j}{l_j}$, we know that $y_j = \frac{l_j a_1}{l_1}$.

Specifically then the point will be

$$\boldsymbol{y} = (y_1, \ldots, y_d) = \left(a_1, \frac{l_2 a_1}{l_1}, \ldots, \frac{l_j a_1}{l_1}, \ldots, \frac{l_d a_1}{l_1}\right)$$

.

The distance of this point $\boldsymbol{y}$ from $\mathbf{0}$ is given by

$$\|\boldsymbol{y} - \mathbf{0}\|^2 = \sum_{j=2}^{d} \frac{l_j^2 a_1^2}{l_1^2} + a_1^2 = a_1^2 \frac{\sum_{j=1}^{d} l_j^2}{l_1^2} = \frac{a_1^2}{l_1^2}.$$

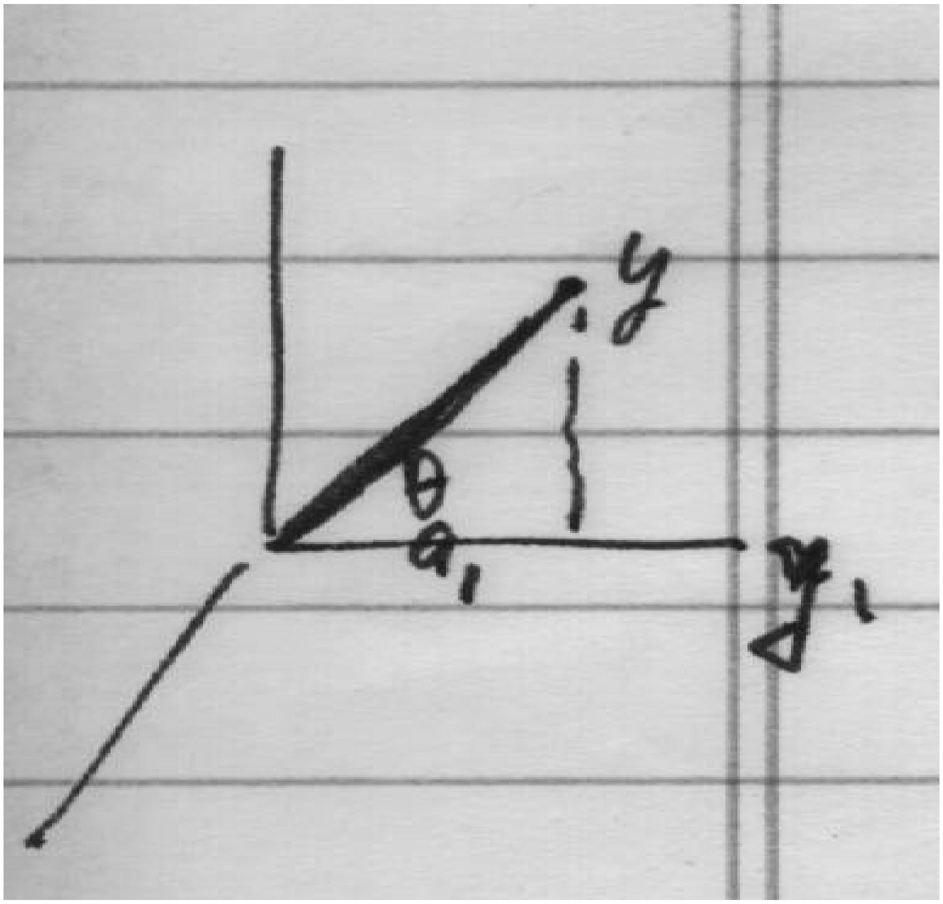
Alternatively, we may write

$$l_1 = \frac{a_1}{\|\boldsymbol{y}\|}.$$

Now recall from the theory of finite dimensional vector spaces, the dot product of two vectors, say $\boldsymbol{x}$ and $\boldsymbol{y}$ has the equation $\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$ cos(θ), where θ is the angle between the two vectors.

In particular if $\boldsymbol{x} = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$, then $\boldsymbol{e}_1 \cdot \boldsymbol{y} = a_1 = \|\boldsymbol{y}\|cos(\theta)$. Solving for cos$(\theta)$ yields

$$\cos(\theta) = \frac{a_1}{\|\boldsymbol{y}\|} = l_1.$$



Thus l_1 has the interpretation of a cosine of the angle between the line, $\frac{y_1}{l_1} = \frac{y_2}{l_2} = \cdots = \frac{y_d}{l_d}$, through the origin and the $(d-1)$-plane given by $y_1 = a_1$. For this reason, the l_j are called the *direction cosines*. Notice if all the direction cosines are equal, say to l, then

$$\sum_{i=1}^{d} l_i^2 = d\, l^2 = 1 \Rightarrow l^2 = \frac{1}{d} \Rightarrow l = \frac{1}{\sqrt{d}}$$

as before in Example 1.2.

Proposition 2.1 The orthogonal projection of a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ onto a line, $\mathcal{L}$: $\frac{y_1}{l_1} = \frac{y_2}{l_2} = \cdots = \frac{y_d}{l_d}$ is a distance $\sum_{i=1}^{d} l_i x_i$ from the origin **0**.

Proof: Exercise for the student.

Proposition 2.2 If a second line has direction cosines l_i', then ϕ is the angle between the two lines where $\cos(\phi) = \sum_{i=1}^{d} l_i l_i'$. The lines are orthogonal if $\sum_{i=1}^{d} l_i l_i' = 0$.

Proof: Exercise for the student.

Now consider a $(d-1)$-flat which is assumed to go through the origin. If so it must have the equation $\sum_{i=1}^{d} a_i y_i = 0$. A line through the origin will have the form

$$\frac{y_1}{l_1} = \frac{y_2}{l_2} = \cdots = \frac{y_d}{l_d} = \rho$$

or, in general, $y_i = \rho\, l_i,\ i = 1, \ldots, d$.

The line will lie on the plane if and only if

$$\sum_{i=1}^{d} a_i y_i = \sum_{i=1}^{d} a_i \rho\, l_i = 0.$$

Since $\rho \neq 0$ except in the trivial case, the line will lie in the plane if and only if $\sum_{i=1}^{d} a_i l_i = 0$.

Thus we know $\sum_{i=1}^{d} a_i l_i = 0$.

Now consider a line whose direction cosines are proportional to a_i. If so then $l'_i = ka_i$, and we have $\sum_{i=1}^{d} \frac{l'_i}{k} l_i = 0$. Since $k \neq 0$ except in the trivial case, $\sum_{i=1}^{d} l'_i l_i = 0$. It follows immediately from Proposition 2.2 that a line with direction cosines proportional to the a_i will be orthogonal to any line in the $(d-1)$-flat.

Proposition 2.3 If $\sum_{i=1}^{d} l_i y_i = p$ is the equation of an arbitrary $(d-1)$-flat, then the l's are the direction cosines of the normal and p is the length of the perpendicular from the origin to the flat.

Proof: The first part of this result follows from our discussion just above. The second part follows from Proposition 2.1.

Angle between two $(d-1)$-flats

The angle between a $(d-1)$-flat and a line is the compliment of the angle between the line and the normal to the $(d-1)$-flat. The angle between two $(d-1)$-flats is defined as the angle between their normals. Thus if $\sum_{i=1}^{d} l_i y_i = p$ and $\sum_{i=1}^{d} l'_i y_i = p'$, then

$$\cos(\phi) = \sum_{i=1}^{d} l_i l'_i \Rightarrow \phi = arccos\Big(\sum_{i=1}^{d} l_i l'_i\Big).$$

Angles between Other Flats

We have now discussed angles between S_1 and S_{d-1}. However, for flats of other dimensions, there are in general more than one angle between them. For example, in S_4 consider two 2-planes. They have a common point. If one plane is fixed, the other may vary in a doubly infinite number of ways. Two additional points or, equivalently, two additional angles are required to fix it.

Consider S_{d-p} defined by

$$a_{i1}y_i + \cdots + a_{id}y_d = 0,\ i = 1, 2, \ldots, p.$$

Let $\boldsymbol{A}$ be the $p \times d$ matrix of coefficients a_{ij} and $\boldsymbol{Y}$ the $d \times 1$ column vector of y's. Then $\boldsymbol{AY} = \mathbf{0}$.

Suppose S_{d-p} goes through the origin. If $\boldsymbol{L}$ is a column vector representing a line in this plane, since $\sum_{i=1}^{d} a_{ij}\, l_j = 0$ and $\sum_{i=1}^{d} l_j^2 = 1$, we have $\boldsymbol{AL} = \mathbf{0}$ and $\boldsymbol{L^T L} = \mathbf{1}$.

Similarly for a second space, S_{d-q}, we have $\boldsymbol{BM} = \mathbf{0}$ and $\boldsymbol{M^T M} = \mathbf{1}$ where $\boldsymbol{M}$ represents a line with direction cosines m_i. Here $\boldsymbol{B}$ has the obvious interpretation analogous to $\boldsymbol{A}$.

The angle between the line in one space and the line in the other is ϕ where

$$\cos(\phi) = \boldsymbol{L^T M = M^T L} = R.$$

Since the angle is maximized when R is minimized, we want to find minimum values of R subject to $\boldsymbol{AL = 0}$, $\boldsymbol{L^T L = 1}$, $\boldsymbol{BM = 0}$, $\boldsymbol{M^T M = 1}$.

We use some ideas from the calculus of variations. We construct Lagrange multipliers $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$, α_1 and α_2. $\boldsymbol{\lambda_1}$ is a $1 \times p$ vector and $\boldsymbol{\lambda_2}$ is a $1 \times q$ vector. Thus we wish to optimize (minimize)

$$\boldsymbol{L^T M} - \boldsymbol{\lambda_1 AL} - \boldsymbol{\lambda_2 BM} - \alpha_1(\boldsymbol{L^T L} - \mathbf{1}) - \alpha_2(\boldsymbol{M^T M} - \mathbf{1}). \tag{2.9}$$

Differentiating (2.9) by l_i and by m_i respectively yields

$$\boldsymbol{M^T} - \boldsymbol{\lambda_1 A} - 2\alpha_1 \boldsymbol{L^T} = 0 \tag{2.10}$$

and

$$\boldsymbol{L^T} - \boldsymbol{\lambda_2 B} - 2\alpha_2 \boldsymbol{M^T} = 0. \qquad (2.11)$$

Post multiply (2.10) by $\boldsymbol{L}$ to yield

$$\boldsymbol{M^T L} - \boldsymbol{\lambda_1 A L} - 2\alpha_1 \boldsymbol{L^T L} = 0$$

$$\boldsymbol{M^T L} - 2\alpha_1 = 0$$

or

$$2\alpha_1 = \boldsymbol{M^T L} = R.$$

Similarly, post multiplying (2.11) by $\boldsymbol{M}$ yields

$$\boldsymbol{L^T M} - \boldsymbol{\lambda_2 B M} - 2\alpha_2 \boldsymbol{M^T M} = 0$$

$$\boldsymbol{L^T M} - 2\alpha_2 = 0$$

or

$$2\alpha_2 = \boldsymbol{L^T M} = R.$$

Thus

$$-\boldsymbol{\lambda_1 A} = 2\alpha_1 \boldsymbol{L^T} - \boldsymbol{M^T} = R\boldsymbol{L^T} - \boldsymbol{M^T} \quad (2.12)$$

and

$$-\boldsymbol{\lambda_2 B} = 2\alpha_2 \boldsymbol{M^T} - \boldsymbol{L^T} = R\boldsymbol{M^T} - \boldsymbol{L^T} \quad (2.13)$$

Post multiply (2.12) by $\boldsymbol{A^T}$ to obtain

$$-\boldsymbol{\lambda_1 A\, A^T} = R\boldsymbol{L^T A^T} - \boldsymbol{M^T A^T}$$

But $\boldsymbol{AL = 0} \Rightarrow \boldsymbol{L^T A^T = 0} \Rightarrow$

$$-\boldsymbol{\lambda_1 A\, A^T} = -\boldsymbol{M^T A^T} \qquad (2.14)$$

Similarly post multiplying (2.12) by $\boldsymbol{B^T}$ yields

$$-\boldsymbol{\lambda_1 A B^T} = R\boldsymbol{L^T B^T}. \qquad (2.15)$$

Finally post multiplying (2.13) by $\boldsymbol{B^T}$ and by $\boldsymbol{A^T}$ yields

$$-\boldsymbol{\lambda_2 B B^T} = -\boldsymbol{L^T B^T} \qquad (2.16)$$

and

$$-\boldsymbol{\lambda_2 B A^T} = R\boldsymbol{M^T A^T} \qquad (2.17)$$

Solving (2.15) and (2.16) simultaneously and (2.14) and (2.17) simultaneously yields

$$R\boldsymbol{\lambda_1 A A^T} + \boldsymbol{\lambda_2 B A^T} = 0$$

and

$$\boldsymbol{\lambda_1 B A^T} + R\boldsymbol{\lambda_2 B B^T} = 0.$$

$\boldsymbol{A}$ is $p \times d$, $\boldsymbol{A^T}$ is $d \times p$ and $\boldsymbol{AA^T}$ is $p \times p$.

Since $(\boldsymbol{AA^T})^{\boldsymbol{T}} = \boldsymbol{AA^T}$, $\boldsymbol{AA^T}$ is symmetric call it $\boldsymbol{U}$.

Similarly $\boldsymbol{B}$ is $q \times d$, $\boldsymbol{A^T}$ is $d \times p$, $\boldsymbol{BA^T}$ is $q \times p$. call it $\boldsymbol{V}$.

Finally, $\boldsymbol{BB^T}$ is $q \times q$ and symmetric, call it $\boldsymbol{W}$.

Thus we have

$$R\boldsymbol{\lambda_1 U} + \boldsymbol{\lambda_2 V} = 0 \text{ and } \boldsymbol{\lambda_1 V^T} + R\boldsymbol{\lambda_2 W} = 0. \quad (2.18)$$

Assuming $p < q$, we can solve these simultaneously to yield

$$R^2\boldsymbol{\lambda_1 U} - \boldsymbol{\lambda_1 V^T W^{-1} V} = \boldsymbol{0}$$

or in determinant form

$$\left| R^2\boldsymbol{U} - \boldsymbol{V^T W^{-1} V} \right| = 0.$$

This is a $p \times p$ determinant in R^2. Solutions for angles are positive and there are p of them.

Example (Numerical) 2.7 Consider S_4 and two planes given as follows.

Plane 1 (S_2) $\quad y_1 + 7y_2 + y_3 = 0$, $\quad y_4 = 0$

Plane 2 (also an S_2) $\quad y_1 + y_2 = 0$, $\quad y_3 + y_4 = 0$

Then $\boldsymbol{A} = \begin{pmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

$$\boldsymbol{U} = \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \begin{pmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 7 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 51 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{W} = \boldsymbol{B}\boldsymbol{B}^{\boldsymbol{T}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\boldsymbol{W}^{-\boldsymbol{1}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and

$$\boldsymbol{V^T} = \boldsymbol{AB^T} = \begin{pmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 0 & 1 \end{pmatrix}$$

Thus

$$\left| R^2 \begin{pmatrix} 51 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 8 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 1 & 1 \end{pmatrix} \right| = 0.$$

$$\left| R^2 \begin{pmatrix} 51 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 32\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right| = 0.$$

Thus we have

$$\begin{vmatrix} 51R^2 - 32\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & R^2 - \frac{1}{2} \end{vmatrix} = 0$$

$$(51R^2 - 32.5)(R^2 - .5) - .25 = 0$$

$$51R^4 - 32.5R^2 - 25.5R^2 + 16.25 - .25 = 0$$

$$51R^4 - 58R^2 + 16 = 0$$

$$(17R^2 - 8)(3R^2 - 2) = 0$$

$R = \pm\sqrt{\frac{8}{17}}$ or $R = \pm\sqrt{\frac{2}{3}}$.

This corresponds to angles

$\theta = 46.686$° and 133.313° (these sum to 180°)

and

$\theta = 35.264$° and 144.735° (these also sum to 180°).