Measure Theory and Linear Spaces

Problem Set 6

1. Let (Ω, Σ, μ) be a measure space and $\mathcal{F}$ be the class of a.e. finite-valued measurable functions on Ω. If f_n, f, $f_0 \in \mathcal{F}$ and for $n \geq 1$ with $f_n \leq f_0$ a.e. and $f_n \to f$ a.e. as $n \to \infty$, then $f \leq f_0$ a.e. Moreover, if the measure space is complete and each f_n is measurable, then so is f. Further, if $f = g$ a.e., then $f_n \to g$ a.e.

2. Let (Ω, Σ, Ω), $\mu(\Omega) < \infty$, be a measure space, and f_n, f, $n \geq 1$, be a.e. finite measurable functions on Ω. Then $f_n \to f$ a.e. $\Rightarrow f_n \xrightarrow{\mu} f$, and $f_n \xrightarrow{\mu} f \Rightarrow f_n \xrightarrow{D} f$. If $f = c$ a.e., a constant, then $f_n \xrightarrow{D} c \Rightarrow f_n \xrightarrow{\mu} c$ also, so that the last two are equivalent if and only if f is a constant.

3. Let μ be an outer measure on Ω, and $f_i : \Omega \to \overline{\mathbb{R}}^+$, $i = 1, 2$ be a pair of functions. Then prove the following:

   (a) $0 \leq f_1 \leq f_2$ a.e. $\Rightarrow 0 \leq \int_\Omega^* f_1 d\mu \leq \int_\Omega^* f_2 d\mu \leq \infty$, and $0 \leq f_1 = a \cdot f_2$ a.e. $\Rightarrow \int_\Omega^* f_1 d\mu = a \cdot \int_\Omega^* f_2 d\mu$ $(0 \leq a < \infty)$.

   (b) $f = \chi_A$, $A \subset \Omega \Rightarrow \int_\Omega^* f d\mu = \mu(A)$; for $f \geq 0$, $\int_\Omega^* f d\mu = 0$ iff $f = 0$ a.e.

   (c) If $0 \leq f$ and $\int_\Omega^* f d\mu < \infty$, then $\{w : f(w) > 0\} \subset \bigcup_{n=1}^\infty A_n$ with $\mu(A) < \infty$ for each $n \geq 1$. If moreover f is not μ-measurable, then each A_n can be chosen μ-measurable so that the support of f is contained in a σ-finite μ-measurable set.

   (d) If $f = f^+ - f^- : \Omega \to \overline{\mathbb{R}}$ is a function and $\int_\Omega^* f d\mu$ is defined then $\int_\Omega^* f d\mu = \int_\Omega^* f^+ d\mu - \int_\Omega^* f^- d\mu$, and

   $$\left| \int_\Omega^* f d\mu \right| \leq \int_\Omega^* |f| d\mu \leq \int_\Omega^* f^+ d\mu + \int_\Omega^* f^- d\mu.$$

   (e) For any $f : \Omega \to \overline{\mathbb{R}}$, $|\int_\Omega^* f d\mu| < \infty$ iff $\int_\Omega^* |f| d\mu < \infty$, and in this case $\int_A^* f d\mu$ is defined for any $A \subset \Omega$.

   (f) In general, the integral is only subadditive: for $f_i : \Omega \to \overline{\mathbb{R}}^+$, $i = 1, 2$,

   $$\int_\Omega^* (f_1 + f_2) d\mu \leq \int_\Omega^* f_1 d\mu + \int_\Omega^* f_2 d\mu$$