Measure Theory and Linear Spaces

Problem Set 5

1. Let K be a closed set of *Lebesgue* measure zero on the unit circle, and let F be any continuous complex-valued function on K. Then there exists a function in A whose restriction to K is F.
2. A commutative *Banach* algebra which is a field is (isomorphic to) the field of complex numbers.
3. Every H^p space $(1 \leq p \leq \infty)$ is (isometrically isomorphic to) the conjugate space of a *Banach* space.
4. A function f in H^∞ is an extreme point of the unit ball in H^∞ if and only if $|f(z)| \leq 1$ and
$$\int_{-\pi}^{\pi} \log[1 - |f(e^{i\theta})|]d\theta = -\infty.$$
5. The closure of the set of extreme points of the unit ball in H^1 consists of all f in H^1 such that
   (a) $\|f\|_1 = 1$;
   (b) f has no zeros in the open unit disc.
6. Let $1 < p < \infty$. Let f be an L^p function on the unit circle with *Fourier* series
$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$
Then
$$\sum_{n=0}^{\infty} c_n e^{in\theta}$$
is the *Fourier* series of an L^p function g, and
$$\|g\|_p \leq K_p \|f\|_p$$
where K_p is a constant depending only on p, not on f. Equivalently, if $\mathbf{h}=\mathbf{u}+\mathbf{iv}$ is in H^p and v vanishes at the origin,
$$\|v\|_p \leq M_p \|u\|_p$$
where M_p is a constant independent of $\mathbf{h}$.
7. For each f in L^∞,
$$\sup_x |\widehat{f}(\phi)| = \|f\|.$$
8. The following statements are all equivalent for $f : \Omega \to \mathbb{R}$ with (Ω, Σ) as a measurable space:

(a) f is measurable,
(b) $\{w : f(w) \leq a\} \in \Sigma$, $a \in \mathbb{R}$,
(c) $\{w : f(w) < a\} \in \Sigma$, $a \in \mathbb{R}$,
(d) $\{w : f(w) \geq a\} \in \Sigma$, $a \in \mathbb{R}$,

9. Let (Ω, Σ) be a measurable space and $f_n : \Omega \to \overline{\mathbb{R}}$ be measurable functions for each $n \geq 1$. Then all the following new function obtained from the f_n are measurable for Σ:

   (a) $af_1 + f_2$, $a \in \mathbb{R}$;
   (b) $f_1 \cdot f_2$;
   (c) $\sup_{n \geq 1} f_n$.

10. Let (Ω, Σ, μ) be a measure space, and f_n, f be a.e. finite-valued functions on Ω. Then we have:

    (a) $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g \Rightarrow f = g$ a.e., and conversely, if $f_n \xrightarrow{\mu} f$, $f = g$ a.e., then $f_n \xrightarrow{\mu} g$.
    (b) $f_n \xrightarrow{\mu} f \Rightarrow f_{n_k} \to f$ a.e. for some subsequence $\{n_k : k \geq 1\}$.
    (c) If $\mu(\Omega) < \infty$, $f_n \to f$ a.e., each f_n is (Σ)-measurable, and Σ is complete then $f_n \xrightarrow{\mu} f$.