Measure Theory and Linear Spaces

Problem Set 4

1. Let $\mathbb{L} = \mathbb{R}^2$. What is the linear manifold generated by two points in the plane $\ni$ the line they determine
   (a) contains the origin?
   (b) does not contain the origin?

2. $$\lim_{n\to\infty} f_n(x) = f(x) \text{ iff } \liminf f_n(x) = f(x) = \limsup f_n(x)$$
   Prove the converse, i.e.
   $$\liminf f_n(x) = f(x) = \limsup f_n(x) \text{ implies } \lim_{n\to\infty} f_n(x) = f(x).$$

3. Prove the following:
   (a) Every neighborhood is an open set.
   (b) E is open iff $E' = \{f \in X : f \notin E\}$ is closed.
   (c) $\overset{\circ}{E}$ is open, $\bar{E}$ is closed.
   E is open iff $E = \overset{\circ}{E}$.
   E is closed iff $E = \bar{E}$. $\overset{\circ}{E} \subset E \subset \bar{E}$
   (d) X and ϕ are both open and closed.
   (X contains all points, so it contains its limit points $\Rightarrow X$ is closed).
   (X is open, given $f \in X$ $S(f, \epsilon) \subset X \Rightarrow X$ is open).
   (e) For any finite family of open sets, $E_1, \cdots, E_n$,
   $$\bigcap_{k=1}^{n} E_k \text{ is open.}$$
   For any finite family of closed sets, $E_1, \cdots, E_n$,
   $$\bigcup_{k=1}^{n} E_k \text{ is closed.}$$
   (f) For any family of open sets
   $\{E_\lambda : \lambda \in \Lambda\}$, the set
   $$\bigcup_{\lambda\in\Lambda} E_\lambda \text{ is open.}$$
   For any family of closed sets
   $\{E_\lambda : \lambda \in \Lambda\}$, the set
   $$\bigcap_{\lambda\in\Lambda} E_\lambda \text{ is closed.}$$

4. Let f and g be measurable on E, then all the following are also measurable on E.

   (a) $f + c$, $\forall c \in \mathbb{R}$.
   (b) $f - g$.
   (c) $c \cdot f$.
   (d) $f + g$.
   (e) $|f|$.
   (f) f^2.
   (g) $f \cdot g$.
   (h) $\frac{1}{g}$, if $g(x) \neq 0 \ \forall x \in E$.
   (i) $\frac{f}{g}$, if $g(x) \neq 0 \ \forall x \in E$.